Bull. Aust. Math. Soc. **103** (2021), 96–103
doi:10.1017/S0004972720000337

THE SECOND MINIMUM/MAXIMUM VALUE OF THE NUMBER OF CYCLIC SUBGROUPS OF FINITE p-GROUPS

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(Received 12 February 2020; accepted 27 February 2020; first published online 20 April 2020)

Abstract

Let $C(G)$ be the poset of cyclic subgroups of a finite group G and let $\mathscr{P}$ be the class of p-groups of order p^n ($n \geq 3$). Consider the function $\alpha : \mathscr{P} \longrightarrow (0, 1]$ given by $\alpha(G) = |C(G)|/|G|$. In this paper, we determine the second minimum value of α, as well as the corresponding minimum points. Since the problem of finding the second maximum value of α has been solved for $p = 2$, we focus on the case of odd primes in determining the second maximum.

2010 *Mathematics subject classification*: primary 20D60; secondary 20D15, 20D25.

Keywords and phrases: number of cyclic subgroups, finite p-groups, exponent of a group.

1. Introduction

Let $\mathscr{P}$ be the class of p-groups of order p^n, where $n \geq 3$ is an integer. Let $G \in \mathscr{P}$ and denote by $C(G)$ its poset of cyclic subgroups. For $k \in \{0, 1, \ldots, n\}$, denote by $c_k(G)$ the number of cyclic subgroups of order p^k of G. There are many results concerning the number of subgroups of G, most of them having connections with Sylow's theorems (see [2, 4]). Related to the quantities $c_k(G)$, Miller proved in [10] (see also [2, Theorem 1.10]) that $c_k(G) \equiv 0 \pmod p$, for all $k \in \{2, 3, \ldots, n\}$, whenever p is an odd prime number. In [11], the same author showed that the only groups $G \in \mathscr{P}$ satisfying $c_k(G) = p$, for all $k \in \{2, 3, \ldots, n-1\}$, are the abelian group $C_p \times C_{p^{n-1}}$ and the modular p-group

$$M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, x^y = x^{1+p^{n-2}} \rangle,$$

the result being valid only for odd primes.

Recently, Garonzi and Lima [5] introduced the ratio $\alpha(G) = |C(G)|/|G|$ for a finite group G. They completely solved some maximum problems, determining the maximum value of $\alpha(G)$, when G belongs to one of the classes of nonabelian,

The first author was supported by the European Social Fund, through Operational Programme Human Capital 2014-2020, Project No. POCU/380/6/13/123623. The second author was supported by NSF of China, Grant No. 11561021.

nonnilpotent, nonsupersolvable and nonsolvable groups and they identified the maximum points in each case. We are also interested in this ratio, but we will view it as a function

$$\alpha : \mathscr{P} \longrightarrow (0, 1] \quad \text{given by } \alpha(G) = \frac{|C(G)|}{|G|}.$$

In [9, Lemma 1], Lazorec and Tărnăuceanu proved that

$$\alpha(G) \leq \alpha(C_p^n) \quad \text{for all } G \in \mathscr{P}.$$

The result can be improved by stating that the equality holds if and only if $\exp(G) = p$. In [13], Richards showed that a finite group of order n has at least $\tau(n)$ cyclic subgroups and the equality holds if and only if the group is isomorphic to C_n. A generalisation of this result was given by Jafari and Madadi in [6, Theorem 2.5]. From Richards' result in [13],

$$\alpha(G) \geq \alpha(C_{p^n}) \quad \text{for all } G \in \mathscr{P} \quad \text{and} \quad \alpha(G) = \alpha(C_{p^n}) \Longleftrightarrow G \cong C_{p^n}.$$

In other words, we know the minimum and the maximum values of α and when equality occurs.

In [1, 8, 12, 16], the authors deduced the second minimum/maximum value of the number of subgroups over the class $\mathscr{P}$. The aim of this paper is to provide some answers to a similar question that can be formulated as: 'What is the second minimum/maximum value of α and what can be said about the minimum/maximum points?' Equivalently, 'What is the second minimum/maximum value of the number of cyclic subgroups over the class $\mathscr{P}$ and what can be said about the minimum/maximum points?' We investigate this question in the following two sections.

2. The second minimum value of α

Firstly, we recall two well-known classifications of certain finite p-groups. The first concerns the finite p-groups having a cyclic maximal subgroup, while the other clsssifies the p-groups of order p^4, where $p \geq 5$.

THEOREM 2.1 [15, Vol. II, Theorem 4.1]. *Let $G \in \mathscr{P}$ such that G possesses a cyclic maximal subgroup. If p is odd, then $G \cong C_p \times C_{p^{n-1}}$ or $G \cong M(p^n)$, while, if $p = 2$, then $G \cong C_2 \times C_{2^{n-1}}$ or G is isomorphic to one of the following nonabelian groups:*

(1) $M(2^n)$, $n \geq 4$;
(2) *the dihedral* 2*-group* $D_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{-1}\rangle$;
(3) *the generalised quaternion group* $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1}\rangle$;
(4) *the quasi-dihedral group* $QD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{2^{n-2}-1}\rangle$, $n \geq 4$.

THEOREM 2.2 ([3, Section 73]; [14, Theorem 1.6.1]). *Let $G \in \mathscr{P}$ such that $p \geq 5$ and $n = 4$. If G is abelian, then G is isomorphic to $C_{p^4}, C_p \times C_{p^3}, C_{p^2} \times C_{p^2}, C_{p^2} \times C_p^2$ or C_p^4. Otherwise, G is isomorphic to one of the following groups:*

(1) $M(p^4)$;
(2) $G_1 = \langle a, b \mid a^{p^2} = b^{p^2} = 1, ba = a^{1+p}b\rangle$;

(3) $G_2 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, cb = a^p bc, ab = ba, ac = ca\rangle$;

(4) $G_3 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ca = a^{1+p} c, ba = ab, cb = bc\rangle$;

(5) $G_4 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ca = abc, ab = ba, bc = cb\rangle$;

(6) $G_5 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ba = a^{1+p} b, ca = abc, cb = bc\rangle$;

(7) $G_6 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, ba = a^{1+p} b, ca = a^{1+p} bc, cb = a^p bc\rangle$;

(8) $G_7 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1,\ ba = a^{1+p} b,\ ca = a^{1+dp} bc,\ cb = a^{dp} bc,\ p \nmid d, p \nmid d-1\rangle$;

(9) $G_8 = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1,\ dc = acd,\ bd = db,\ ad = da,\ bc = cb, ac = ca,\ ab = ba\rangle$;

(10) $G_9 = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1,\ dc = bcd,\ db = abd,\ ad = da,\ bc = cb, ac = ca,\ ab = ba\rangle$.

Using [9, Theorem 2], one can compute the number of cyclic subgroups of any finite abelian p-group. By applying the Inclusion-Exclusion Principle, the number of cyclic subgroups of each of the nonabelian p-groups possessing a cyclic maximal subgroup was determined in [17]. Also, in [14, Section 2.3.2], one can find the number of cyclic subgroups for each of the nonabelian groups listed in Theorem 2.2. To summarise,

$$\begin{cases} |C(C_p \times C_{p^{n-1}})| = |C(M(p^n))| = (n-1)p + 2 \\ |C(D_{2^n})| = 2^{n-1} + n, \quad |C(Q_{2^n})| = 2^{n-2} + n, \quad |C(QD_{2^n})| = 3 \cdot 2^{n-3} + n \\ |C(C_{p^4})| = 5, \quad |C(C_{p^2} \times C_{p^2})| = |C(G_1)| = p^2 + 2p + 2 \\ |C(C_{p^2} \times C_p^2)| = |C(G_i)| = 2p^2 + p + 2 \quad \text{for } i \in \{2, 3, 4, 5, 6, 7\} \\ |C(C_p^4)| = |C(G_i)| = p^3 + p^2 + p + 2 \quad \text{for } i \in \{8, 9\}. \end{cases} \tag{2.1}$$

Finally, we recall that the number of cyclic subgroups of a finite group G is

$$|C(G)| = \sum_{x \in G} \frac{1}{\varphi(o(x))},$$

where φ is Euler's totient function and $o(x)$ is the order of an element x of G. Before stating the main result of this section, we prove the following preliminary fact.

LEMMA 2.3. *Let $G \in \mathscr{P}$ such that $n \geq 4, G \not\cong C_{p^n}$ and* $\exp(G) \leq p^{n-2}$. *Then*

$$|C(G)| > |C(M(p^n))|.$$

PROOF. Let $G \in \mathscr{P}$ satisfy the hypotheses of the lemma. We proceed by induction on $|G|$. For the initial step, if $|G| \in \{16, 81\}$, the result holds and may be checked using GAP [18]. Also, if G is a p-group of order p^4, where $p \geq 5$ and $\exp(G) \in \{p, p^2\}$, the conclusion easily follows by means of the results in (2.1).

For the inductive step, let N be a maximal subgroup of G. Obviously $N \not\cong C_{p^{n-1}}$ since $\exp(G) \leq p^{n-2}$. If $\exp(N) \leq p^{n-3}$, the inductive hypothesis and (2.1) imply that

$$\begin{aligned}|C(G)| &= |C(N)| + \sum_{x \in G \setminus N} \frac{1}{\varphi(o(x))} \\ &> |C(M(p^{n-1}))| + \sum_{x \in G \setminus N} \frac{1}{\varphi(p^{n-2})} \\ &= (n-2)p + 2 + p^2 > |C(M(p^n))|.\end{aligned}$$

If $\exp(N) = p^{n-2}$, then N has a cyclic maximal subgroup. From Theorem 2.1 and (2.1), it follows that $|C(N)| \geq |C(M(p^{n-1}))|$ and so,

$$\begin{aligned}|C(G)| = |C(N)| + \sum_{x \in G \setminus N} \frac{1}{\varphi(o(x))} &\geq |C(M(p^{n-1}))| + \sum_{x \in G \setminus N} \frac{1}{\varphi(p^{n-2})} \\ &\geq (n-2)p + 2 + p^2 > |C(M(p^n))|,\end{aligned}$$

which completes the proof. □

We are ready to prove the main result of this section which determines the second minimum value of α.

THEOREM 2.4. *Let $G \in \mathscr{P}$ such that $G \not\cong C_{p^n}$.*

(i) *If p is odd, then $\alpha(G) \geq \alpha(C_p \times C_{p^{n-1}}) = \alpha(M(p^n))$.*

(ii) *If $p = 2$, then* $\begin{cases} \alpha(G) \geq \alpha(Q_8) & \text{if } n = 3, \\ \alpha(G) \geq \alpha(C_2 \times C_8) = \alpha(M(16)) = \alpha(Q_{16}) & \text{if } n = 4, \\ \alpha(G) \geq \alpha(C_2 \times C_{2^{n-1}}) = \alpha(M(2^n)) & \text{if } n \geq 5. \end{cases}$

Equality holds if and only if G is isomorphic to one of the indicated minimum points corresponding to each case.

PROOF. Let $G \in \mathscr{P}$ such that $G \not\cong C_{p^n}$. Once again, we reason by induction on $|G|$. For the initial step, one may use GAP to check the above result for $|G| \in \{8, 16\}$. Also, for an odd prime p and $n = 3$, the classification of finite noncyclic p-groups of order p^3 is well-known (see [15, Vol. II, (4.13)]). These groups are $C_p \times C_{p^2}$, C_p^3, $M(p^3)$ and $E(p^3)$, where

$$E(p^3) = \langle x, y \mid x^p = y^p = [x, y]^p = 1, [x, y] \in Z(E(p^3)) \rangle.$$

Since $\exp(C_p^3) = \exp(E(p^3)) = p$ and, as we mentioned in Section 1, the number of cyclic subgroups attains its maximum value in this case, our result also holds. Consequently, for the inductive step, if $p = 2$, we assume that $n \geq 5$, while, if p is an odd prime, we take $n \geq 4$. Following on from this, it is sufficient to show that

$$|C(G)| \geq |C(C_p \times C_{p^{n-1}})| = |C(M(p^n))|$$

and that the equality holds if and only if $G \cong C_p \times C_{p^{n-1}}$ or $G \cong M(p^n)$.

Let N be a maximal subgroup of G. If $N \ncong C_{p^{n-1}}$, from the inductive hypothesis and (2.1),

$$\begin{aligned}|C(G)| &= |C(N)| + \sum_{x\in G\setminus N} \frac{1}{\varphi(o(x))}\\ &\geq |C(C_p \times C_{p^{n-2}})| + \sum_{x\in G\setminus N} \frac{1}{\varphi(p^{n-1})}\\ &= |C(C_p \times C_{p^{n-1}})| = |C(M(p^n))|.\end{aligned}$$

If $N \cong C_{p^{n-1}}$, then G has a cyclic maximal subgroup, so G is isomorphic to one of the groups listed in Theorem 2.1 and by (2.1), $|C(G)| \geq |C(C_p \times C_{p^{n-1}})| = |C(M(p^n))|$.

To study the equality case, suppose that $|C(G)| = |C(C_p \times C_{p^{n-1}})| = |C(M(p^n))|$. By Lemma 2.3, $\exp(G) = p^{n-1}$, so G is isomorphic to one of the groups listed in Theorem 2.1. Taking into consideration the defining equality for this case, it follows that $G \cong C_p \times C_{p^{n-1}}$ or $G \cong M(p^n)$. Since the converse is true, the proof is finished. □

3. The second maximum value of α

Let $G \in \mathscr{P}$. We consider the set $\Omega_{\{1\}}(G) = \{x \in G \mid x^p = 1\}$ and we denote by $\Omega_1(G)$, the first omega subgroup of G, that is, $\Omega_1(G) = \langle \Omega_{\{1\}}(G) \rangle$.

If $\exp(G) \neq p$ and $p = 2$, then [5, Corollaries 2 and 3] (see also [16, Lemma 1.4]), lead to

$$\alpha(G) \leq \alpha(D_8 \times C_2^{n-3}) \quad \text{and} \quad \alpha(G) = \alpha(D_8 \times C_2^{n-3}) \Longleftrightarrow G \cong D_8 \times C_2^{n-3}.$$

Following on from this, the second maximum value of α, as well as the maximum points, are known for $p = 2$. In what follows, we are interested in the case of odd primes and we solve the second maximum problem while working under the assumption that $\Omega_1(G) \neq G$.

Theorem 3.1. *Let p be an odd prime and $G \in \mathscr{P}$ such that $\exp(G) \neq p$. If $\Omega_1(G) \neq G$, then*

$$\alpha(G) \leq \frac{2p^{n-2} + p^{n-3} + \cdots + p + 2}{p^n},$$

and equality holds if and only if $\exp(G) = p^2$ and $\Omega_{\{1\}}(G) = \Omega_1(G)$ is of index p in G.

Proof. Let $\exp(G) = p^m$, where $m \geq 2$ is an integer. Then

$$\begin{aligned}p^n &= 1 + (p-1)c_1(G) + (p^2 - p)c_2(G) + \cdots + (p^m - p^{m-1})c_m(G)\\ &\geq 1 + (p-1)c_1(G) + (p^2 - p)(c_2(G) + c_3(G) + \cdots + c_m(G))\\ &= 1 + (p-1)c_1(G) + (p^2 - p)(|C(G)| - 1 - c_1(G)),\end{aligned}$$

so

$$|C(G)| \leq \frac{p^n + p^2 - p - 1 + (p-1)^2 c_1(G)}{p^2 - p}, \tag{3.1}$$

and the equality holds if and only if $\exp(G) = p^2$.

Since $\Omega_1(G) \neq G$, it follows that $[G : \Omega_1(G)] \geq p$ and therefore,

$$|\Omega_{\{1\}}(G)| \leq |\Omega_1(G)| \leq p^{n-1}, \tag{3.2}$$

which leads to

$$c_1(G) \leq \frac{p^{n-1} - 1}{p - 1}.$$

Using (3.1) and the above inequality,

$$|C(G)| \leq \frac{p^n + p^2 - p - 1 + (p-1)(p^{n-1} - 1)}{p^2 - p} = 2p^{n-2} + p^{n-3} + \cdots + p + 2,$$

and the conclusion follows. Moreover, equality holds if and only if we have equality in (3.1) and (3.2), that is, $\exp(G) = p^2$ and $\Omega_{\{1\}}(G) = \Omega_1(G)$ is of index p in G. □

Two examples of the equality case of Theorem 3.1 are the groups $M(p^3) \times G_1$, where $|G_1| = p^{n-3}$ and $\exp(G_1) = p$ and $C_{p^2} \times G_2$, where $|G_2| = p^{n-2}$ and $\exp(G_2) = p$.

Remark 3.2. Under the same hypotheses as in Theorem 3.1, what can be said if $\Omega_1(G) = G$?

First of all, the result does not hold in this case. For instance, if we take $p = 3$ and $n = 4$, the wreath product $C_3 \wr C_3$ (SmallGroup(81,7)) and the group $E(27) \rtimes C_3$ (SmallGroup(81,9)) satisfy $\Omega_1(G) = G$ and they have 29 and 35 cyclic subgroups, respectively. However, the group $M(27) \times C_3$ (SmallGroup(81,13)) has only 23 cyclic subgroups and it satisfies the equality case of Theorem 3.1. Moreover, $\alpha(G) \leq \alpha(E(27) \rtimes C_3)$, while the equality holds if and only if $G \cong E(27) \rtimes C_3$.

Secondly, if $\Omega_1(G) = G$, then G can be generated by elements of order p. Consequently, G/G' can be generated by elements of order p, so G/G' is elementary abelian and hence $\Phi(G) \subseteq G'$ and, since the converse inclusion is well known, we obtain $G' = \Phi(G)$. If $G' \subseteq Z(G)$, then G is a nilpotent group of class 2 and, since p is odd, it is known that $\exp(\Omega_1(G)) = p$. This implies $\exp(G) = p$, which is a contradiction. To completely solve the proposed maximum problem, a first approach would be to classify the groups $G \in \mathscr{P}$ satisfying $G' = \Phi(G) \nsubseteq Z(G)$ and compare their numbers of cyclic subgroups. A second option would be to find the maximum number of solutions of $x^p = 1$ in G. This was done in [7] for $p = 3$, giving

$$2c_1(G) \leq 7 \cdot 3^{n-2} - 1,$$

and the equality case holds here if and only if $G \cong E(27) \rtimes C_3^2$ (SmallGroup(243,58)) or $G \cong C_3 \times E(27) \rtimes C_3$ (SmallGroup(243,53)). Substituting in (3.1),

$$|C(G)| \leq \frac{23 \cdot 3^{n-3} + 1}{2},$$

so

$$\alpha(G) \leq \frac{23 \cdot 3^{n-3} + 1}{2 \cdot 3^n}.$$

The upper bound is attained if and only if one works with a group G isomorphic to $(E(27) \rtimes C_3^2) \times G_1$ or to $(C_3 \times E(27) \rtimes C_3) \times G_1$, where $|G_1| = 3^{n-5}$ and $\exp(G_1) = 3$.

We end our paper by suggesting the following open problem.

Problem 3.3. Let $p \geq 5$ be a prime number and let $G \in \mathscr{P}$ such that $\Omega_1(G) = G$. Determine the second maximum value of α in this case.

A solution to this problem along with our results would completely determine the second maximum value of α.

Acknowledgement

The authors are grateful to the reviewer for remarks that improved the previous version of the paper.

References

[1] S. Aivazidis and T. Müller, 'Finite non-cyclic p-groups whose number of subgroups is minimal', *Arch. Math. (Basel)* **114**(1) (2020), 13–17.
[2] Y. Berkovich, *Groups of Prime Power Order,* Vol. 1 (de Gruyter, Berlin, 2008).
[3] W. Burnside, *Theory of Groups of Finite Order*, 2nd edn (Dover Publications, Inc., New York, 1955).
[4] L. M. Butler, 'Subgroup lattices and symmetric functions', *Mem. Amer. Math. Soc.* **112** (1994), 1–160.
[5] M. Garonzi and I. Lima, 'On the number of cyclic subgroups of a finite group', *Bull. Braz. Math. Soc.* **49** (2018), 515–530.
[6] M. H. Jafari and A. R. Madadi, 'On the number of cyclic subgroups of a finite group', *Bull. Korean Math. Soc.* **54**(60) (2017), 2141–2147.
[7] T. J. Laffey, 'The number of solutions of $x^3 = 1$ in a 3-group', *Math. Z.* **149**(2) (1976), 43–45.
[8] M. S. Lazorec, 'A connection between the number of subgroups and the order of a finite group', Preprint, 2019, arXiv:1901.06425.
[9] M. S. Lazorec and M. Tărnăuceanu, 'A note on the number of cyclic subgroups of a finite group', *Bull. Math. Soc. Sci. Math. Roumanie* **62/110**(4) (2019), 403–416.
[10] G. A. Miller, 'An extension of Sylow's theorem', *Proc. Lond. Math. Soc. Ser. 2* **2** (1905), 142–143.
[11] G. A. Miller, 'The groups of order p^m which contain exactly p cyclic subgroups of order p^α', *Trans. Amer. Math. Soc.* **7**(2) (1906), 228–232.
[12] H. Qu, 'Finite non-elementary abelian p-groups whose number of subgroups is maximal', *Israel J. Math.* **195** (2013), 773–781.
[13] I. M. Richards, 'A remark on the number of cyclic subgroups of a finite group', *Amer. Math. Monthly* **91**(9) (1984), 571–572.
[14] G. Sædén Ståhl, J. Laine and G. Behm, *On p-groups of Low Power Order*, Bachelor Thesis, KTH Royal Institute of Technology, 2010.
[15] M. Suzuki, *Group Theory,* Vols. I, II (Springer, Berlin, 1982, 1986).
[16] M. Tărnăuceanu, 'On a conjecture by Haipeng Qu', *J. Group Theory* **22**(3) (2019), 505–514.
[17] M. Tărnăuceanu and L. Tóth, 'Cyclicity degrees of finite groups', *Acta Math. Hungar.* **145** (2015), 489–504.
[18] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3*, 2018, https://www.gap-system.org.

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